1 What is Amdahl's Law?

Amdahl's law is an expression used to find the maximum expected improvement to an overall system when only part of the system is improved. It is often used in parallel computing to predict the theoretical maximum speedup using multiple processors.

(source: https://en.wikipedia.org/wiki/Amdahl%27s_law)

2 What Kinds of Problems Do We Solve with Amdahl's Law?

Recall how we defined performance of a system that has been sped up:

$$\text{Speedup} = \frac{\text{Execution time before improvement}}{\text{Execution time after improvement}}$$

Amdahl's Law states:

$$\text{Speedup} = \frac{1}{\left(\dfrac{\text{fraction enhanced}}{\text{factor of improvement}}\right) + (1 - \text{fraction enhanced})}$$

Let

S = Speedup

T_{before} = Execution time before improvement

T_{after} = Execution time after improvement

F_e = Fraction enhanced

F_i = Factor of improvement

Then we can rewrite Amdahl's Law as

$$S = \frac{T_{before}}{T_{after}} = \frac{1}{\left(\dfrac{F_e}{F_i}\right) + (1 - F_e)}$$

$$\frac{1}{S} = \frac{T_{after}}{T_{before}} = \left(\frac{F_e}{F_i}\right) + (1 - F_e)$$

There are three types of problems to be solved using Amdahl's Law:

1. Given F_e and F_i, determine S

2. Given F_e and S, determine F_i

3. Given F_i and S, determine F_e

Let us consider an example of each type of problem.

2.1 Problem Type 1 – Predict System Speedup S

If we know F_e and F_i, then we use the Speedup equation above to determine S.

Example: Let a program have 40 percent of its code enhanced ($F_e = 0.4$) to run 2.3 times faster ($F_i = 2.3$). What is the overall system speedup S?

$$
\begin{aligned}
S &= \frac{1}{\left(\frac{F_e}{F_i}\right) + (1 - F_e)} \\
&= \frac{1}{\left(\frac{0.4}{2.3}\right) + (1 - 0.4)} \\
&= \frac{1}{0.174 + 0.6} \\
&= \frac{1}{0.774} \\
&= 1.292
\end{aligned}
$$

So if we make 40% of the code run 2.3 times faster, then the overall system speedup will be 1.292.

2.2 Problem Type 2 – Predict Factor of Improvement F_i

If we know F_e and S, then we can solve the Speedup equation above to determine F_i.

Example: Let a program have 40 percent of its code enhanced ($F_e = 0.4$) to yield a speedup of 1.3 times faster ($S = 1.3$). What is the factor of improvement F_i of the portion enhanced?

First we need to see if it is even possible to do that. In other words, if by enhancing 40 percent of the program, is it possible to make the program run 1.3 times faster?

Assume the limit, where F_i = infinity, so

$$
\begin{aligned}
S &= \frac{1}{\left(\frac{F_e}{F_i}\right) + (1- F_e)} \\
&= \frac{1}{\left(\frac{0.4}{\infty}\right) + (1- 0.4)} \\
&= \frac{1}{0+0.6} \\
&= \frac{1}{0.6} \\
&= 1.667
\end{aligned}
$$

Since we can achieve a maximum speedup S of 1.667, therefore, we cannot achieve the desire speedup of 1.3. Now we can calculate the actual factor of improvement F_i in order to achieve this speedup.

$$
S = \frac{1}{\left(\frac{F_e}{F_i}\right) + (1- F_e)}
$$

$$
\frac{1}{S} = \left(\frac{F_e}{F_i}\right) + (1- F_e) \quad \text{invert both sides}
$$

$$
\frac{1}{S} - (1- F_e) = \left(\frac{F_e}{F_i}\right)
$$

$$
F_i = \frac{F_e}{\frac{1}{S} - (1- F_e)}
$$

First, solve the speedup equation for F_i, and then plug in the values to solve for F_i.

$$
\begin{aligned}
F_i &= \frac{F_e}{\frac{1}{S} - (1- F_e)} \\
&= \frac{0.4}{\frac{1}{1.3} - (1- 0.4)} \\
&= \frac{0.4}{0.769 - 0.6} \\
&= \frac{0.4}{0.169} \\
&= 2.367
\end{aligned}
$$

So if we improve 40% of the program by 2.367 times, then we can achieve a speedup of 1.3 times.

2.3 Problem Type 3 – Predict Fraction of System to be Enhanced F_e

If we know F_i and S, then we can solve the Speedup equation to determine F_e.

Example: Let a program have a portion of its code enhanced F_e to run 4 times faster ($F_i = 4$), to yield a system speedup of 3.3 times faster ($S = 3.3$). What is the fraction enhanced (F_e)?

$$S = \frac{1}{\left(\frac{F_e}{F_i}\right) + (1 - F_e)}$$

$$F_e = \frac{F_i}{S - SF_i} - \frac{F_i}{1 - F_i}$$

Plug the values into the speedup equation and solve for F_e.

$$S = \frac{1}{\left(\frac{F_e}{F_i}\right) + (1 - F_e)}$$

$$3.3 = \frac{1}{\left(\frac{F_e}{4}\right) + (1 - F_e)}$$

$$\frac{1}{3.3} = \left(\frac{F_e}{4}\right) + (1 - F_e) \quad \text{invert both sides}$$

$$0.303 = 1 - F_e + \left(\frac{F_e}{4}\right)$$

$$= 1 - \frac{3}{4}F_e$$

$$0.303 = 1 - 0.75F_e$$

$$0.75F_e = 1 - 0.303$$

$$F_e = \frac{1 - 0.303}{0.75}$$

$$= 0.929$$

So if we enhance 92.9% of the program to run 4 times faster, then the overall system speedup will be 3.3 times faster.